Path integration over the n-dimensional Euclidean group

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The path integral for the n-dimensional free particle is considered. According to the underlying symmetry, the short time propagator is expanded in zonal spherical functions of the Euclidean group $G = T^n \rtimes SO(n)$ with respect to the subgroup $H = SO(n)$. The group theoretical approach to path integration, including the radial part, is explicitly demonstrated.

I. INTRODUCTION

Recently, the present authors[1] have developed a general scheme for path integration on a homogeneous space given by a group quotient G/H, $H \subset G$. For a trivial group $H = \{e\}$ the short time propagator has been expanded in group characters of G. In all other cases $H \neq \{e\}$ the group expansion has led to a decomposition of the short time propagator in zonal spherical functions. Up to now this technique has only been applied to the generalized polar coordinate path integral[1] and to the path integration on spaces with positive and negative curvature.[2] The purpose of the present paper is to include radial path integrals in this group theoretical approach. The free particle in n dimensions is considered where the Euclidean space E_n is viewed as the quotient G/H. Here G is the n-dimensional Euclidean group, which is a semidirect product of the translation group and the rotation group in n dimensions, $T^n \rtimes SO(n)$, and $H = SO(n)$.

This paper is organized as follows. In the next section, the n-dimensional Euclidean group and its representations are discussed in some detail. The Fourier decomposition of functions $f(g)$ of $g \in G$, satisfying $f(h^{-1}gh) = f(g)$ for $h \in H$, is constructed explicitly. In Sec. III this decomposition is applied to expand the short time propagator in zonal spherical functions $D^k_{00}(g)$ of $G \supset H$. An integral representation of the free particle propagator is obtained, leading to the well-known result of Feynman.[3]

II. THE EUCLIDEAN GROUP IN n DIMENSIONS, $G = T^n \rtimes SO(n)$

The Euclidean group $G = T^n \rtimes SO(n)$ acts as a transformation group in the Euclidean space E_n of n dimensions via the map

$$g: \mathbf{a} \mapsto h\mathbf{a} + \mathbf{r}, \quad g \in G, \tag{2.1}$$

where h is an $n \times n$ matrix representation of the subgroup $H = SO(n)$. The parameters of the group element $g = g(\mathbf{r}, h)$ are the $n(n-1)/2$ Euler angles of h and the n coordinates of the translation vector $\mathbf{r}$ given (for convenience) in polar coordinates $(r, \varphi_1, \varphi_2, \ldots, \varphi_{n-1})$. The group composition law is

$$g(\mathbf{r}_1, h_1) g(\mathbf{r}_2, h_2) = g(\mathbf{r}_1 + h_1 \mathbf{r}_2, h_1 h_2). \tag{2.2}$$

A general group element may be decomposed into a translation and a rotation (see Ref. 4, p. 548)

$$g(\mathbf{r}, h) = g(\mathbf{r}, \mathbf{1}) g(\mathbf{o}, h) = g(\mathbf{o}, h) g(h^{-1}\mathbf{r}, \mathbf{1}), \tag{2.3}$$

where $\mathbf{1}$ stands for the $n \times n$ unit matrix and $\mathbf{o}$ is the n-dimensional null vector. Obviously any point $\mathbf{r}$ in E_n may be obtained via a translation of the origin $\mathbf{o}$,

$$g: \mathbf{o} \mapsto \mathbf{r}. \tag{2.4}$$

Accordingly we may restrict g in (2.4) to the form $g(\mathbf{r}, \mathbf{1})$, as the origin is invariant under pure rotations $g(\mathbf{o}, h)$. Moreover, any function $f(\mathbf{r})$ defined over E_n may be viewed as a function $f(g)$ on the group manifold of G. Especially if $f(\mathbf{r}) = f(r)$ depends only on the radial distance r, it is a function invariant under rotations $g(\mathbf{o}, h)$. The zonal spherical functions $D^k_{00}(g)$ having this property are given by Bessel functions (see Ref. 4, p. 553)

$$D^k_{00}(g) = \Gamma(n/2)(2/kr)^{(n-2)/2} J_{(n-2)/2}(kr), \tag{2.5}$$

where r is the radial polar coordinate of the translation vector $\mathbf{r}$ in $g(\mathbf{r}, h)$. The basis states of G are usually labeled by k, l, and M corresponding to the conserved energy ($E = \hbar^2 k^2/2m$), angular momentum, and its degeneracy, respectively.

For a translation by r along a fixed axis $\mathbf{a}$, e.g., the unit vector in x_n-direction, the associate zonal spherical function reads (see Ref. 4, p. 554)

$$D^k_{L0}(g(r\mathbf{a}, \mathbf{1})) = i^l \Gamma(n/2) \left[(2l + n - 2) \frac{\Gamma(l + n - 2)}{l!\, \Gamma(n-1)} \right]^{1/2} \times \left(\frac{2}{kr} \right)^{(n-2)/2} J_{l + (n-2)/2}(kr), \tag{2.6}$$

where L stands for the $(n-1)$-tuple $L = (l, 0, \ldots, 0)$, with $l = 0, 1, 2, \ldots$. Note that any $\mathbf{r}$ may be obtained from $r\mathbf{a}$ through a pure rotation $h \in SO(n)$, $\mathbf{r} = h(r\mathbf{a})$. (See Ref. 1.)

As is known, a function $f(g)$ invariant under a rotation $g(\mathbf{o}, h)$ may be expanded in zonal spherical functions:

$$f(g) = \int_0^\infty dk\, F(k) d_k D^k_{00}(g), \tag{2.7}$$

where

$$F(k) = \int_G dg\, f(g) D^{k*}_{00}(g). \tag{2.8}$$

In the above, dg is the invariant volume element of G given by

$$dg = d\mathbf{r}\, dh, \tag{2.9}$$

where dh is the normalized invariant Haar measure of

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$H=\mathrm{SO}(n)$, $\int_H dh=1$ and $d\mathbf{r}$ is the usual Euclidean measure. The "dimension" d_k is defined by[1]

$$\frac{\delta(k-k')}{d_k}=\int_G dg\, D^{k}_{00}(g)D^{k'}_{00}(g). \qquad (2.10)$$

Using the explicit form (2.5) we find

$$\frac{\delta(k-k')}{d_k}=\Gamma\left(\frac{n}{2}\right)\frac{2^{n-1}\pi^{n/2}}{(kk')^{(n-2)/2}} \times\int_0^\infty dr\, rJ_{(n-2)/2}(kr)J_{(n-2)/2}(k'r). \qquad (2.11)$$

Comparison with the closure relation of Bessel's function[5]

$$\int_0^\infty dr\, rJ_\nu(kr)J_\nu(k'r)=\frac{1}{k}\delta(k-k') \qquad (2.12)$$

leads to the identification

$$d_k=k^{n-1}/[2^{n-1}\pi^{n/2}\Gamma(n/2)]. \qquad (2.13)$$

For $n=2$, d_k agrees with Barut and Raczka[6] who discuss the harmonic analysis of $T^2 \circledS \mathrm{SO}(2)$.

Finally we would like to mention that for $\tilde{f}(r)=f(g)r^{(n-1)/2}(2\pi)^{n/2}$ and $\tilde{F}(k)=F(k)k^{(n-1)/2}$ the transformation (2.7) leads to the Hankel transformation of order $\nu=(n-2)/2$:

$$\tilde{f}(r)=\int_0^\infty dk\,\tilde{F}(k)J_\nu(kr)\sqrt{kr},$$
$$\tilde{F}(k)=\int_0^\infty dr\,\tilde{f}(r)J_\nu(kr)\sqrt{kr}. \qquad (2.14)$$

III. PATH INTEGRATION OVER G

As an application of the above group expansion we consider the Feynman propagator of an n-dimensional free particle given in the sliced time basis

$$K(\mathbf{r}_b,\mathbf{r}_a;T)=\lim_{N\to\infty}\int\prod_{j=1}^{N}\left[\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2}\exp\left\{\left(\frac{i}{\hbar}\right)S_j\right\}\right] \times\prod_{j=1}^{N-1}d\mathbf{r}_j, \qquad (3.1)$$

where the short time action is given by

$$S_j=(m/2\epsilon)(\Delta\mathbf{r}_j)^2. \qquad (3.2)$$

Here we have adopted the usual notation $\Delta\mathbf{r}_j=\mathbf{r}_j-\mathbf{r}_{j-1}$, $\mathbf{r}_a=\mathbf{r}_0$, $\mathbf{r}_b=\mathbf{r}_N$ and an isometric time slicing $\epsilon N=t_b-t_a=T$.

Let us consider the group element $g_j=g(\mathbf{r}_j,1)$. Obviously the origin is mapped onto $\mathbf{r}_j$ via the translation g_j. The combination,

$$g_{j-1}^{-1}g_j=g^{-1}(\mathbf{r}_{j-1},1)g(\mathbf{r}_j,1)=g(\mathbf{r}_j-\mathbf{r}_{j-1},1), \qquad (3.3)$$

is just the translation mapping $\mathbf{o}$ onto $\Delta\mathbf{r}_j$. Therefore the short time propagator in (3.1) may be considered as a function $f(g_{j-1}^{-1}g_j)$ on G which depends only on the parameter $r=|\Delta\mathbf{r}_j|$ of the group element (3.3). Hence the Fourier decomposition (2.7) may be applied to $\exp\{(i/\hbar)S_j\}$, where the coefficient (2.8) is given by ($z=m/2i\hbar\epsilon$),

$$F(k)=2\pi^{n/2}(2/k)^{(n-2)/2} \times\int_0^\infty dr\, r^{n/2}e^{-zr^2}J_{(n-2)/2}(kr). \qquad (3.4)$$

In (3.4) the integration over the subgroup H and the group parameters $(\varphi_1,\dots,\varphi_{n-1})$ of the translation vector $\mathbf{r}$ has been performed. Using the integral formula[7]

$$\int_0^\infty dx\, x^{\nu+1}e^{-\alpha x^2}J_\nu(\beta x)=\beta^\nu(2\alpha)^{-\nu-1}\exp\{-\beta^2/4\alpha\},$$
$$\mathrm{Re}\,\alpha>0,\ \beta>0,\ \mathrm{Re}\,\nu>-1, \qquad (3.5)$$

we find

$$F(k)=(\pi/z)^{n/2}\exp\{-k^2/4z\}. \qquad (3.6)$$

To be more explicit we have derived the decomposition

$$\left(\frac{m}{2\pi i\hbar\epsilon}\right)^{n/2}\exp\left\{\frac{im}{\hbar 2\epsilon}|\Delta\mathbf{r}_j|^2\right\}=\int_0^\infty dk\exp\left\{-\frac{i\hbar k^2\epsilon}{2m}\right\}d_kD^k_{00}(g_{j-1}^{-1}g_j). \qquad (3.7)$$

With the aid of the orthogonality relation,

$$\int d\mathbf{r}_j\, D^k_{00}(g_{j-1}^{-1}g_j)D^{k'*}_{00}(g_j^{-1}g_{j+1})=\frac{\delta(k-k')}{d_k}D^k_{00}(g_{j-1}^{-1}g_{j+1}), \qquad (3.8)$$

the path integration can be performed leading to the following integral representation of the free particle propagator:

$$K(\mathbf{r}_b,\mathbf{r}_a;T)=\int_0^\infty dk\exp\left\{-\frac{i}{\hbar}\frac{\hbar^2k^2}{2m}T\right\}d_kD^k_{00}(g_a^{-1}g_b). \qquad (3.9)$$

The energy spectrum may be identified to be $E_k=\hbar^2k^2/2m$. In order to obtain the normalized wave functions we make use of the group property

$$D^k_{00}(g_a^{-1}g_b)=\sum_L D^k_{L0}(g_b)D^{k*}_{L0}(g_a). \qquad (3.10)$$

As $\mathbf{r}=h(r\mathbf{a})$, it follows from (2.3) that $g=g(\mathbf{r},1)=g(\mathbf{o},h)g(r\mathbf{a},1)g(\mathbf{o},h^{-1})$ and the associate spherical functions

$$D^k_{L0}(g(\mathbf{o},h)g(r\mathbf{a},1)g(\mathbf{o},h^{-1}))=D^k_{L0}(g(\mathbf{o},h)g(r\mathbf{a},1))$$

decompose into

$$D^k_{L0}(g)=\sum_{L'}D^k_{LL'}(g(\mathbf{o},h))D^k_{L'0}(g(r\mathbf{a},1)). \qquad (3.11)$$

Note that the sum vanishes unless L' is of the form $L'=(l,0,\dots,0)$ (see Ref. 4, p. 555) and $D^k_{LL'}(g(\mathbf{o},h))$ reduces to the associate spherical functions of $\mathrm{SO}(n)$, $D^k_{LL'}(g(\mathbf{o},h))=d^l_{M0}(h)$, given in Ref. 1. Collecting everything, the propagator (3.9) is rewritten as

$$K(\mathbf{r}_b,\mathbf{r}_a;T)=\int_0^\infty dk\exp\left\{-\left(\frac{i}{\hbar}\right)E_kT\right\} \times\sum_{l,M}\Psi_{klM}(\mathbf{r}_b)\Psi^*_{klM}(\mathbf{r}_a) \qquad (3.12)$$

with

$$\Psi_{KlM}(\mathbf{r}) = \sqrt{d_k}\, D^{k}_{L'0}(g(r\mathbf{a},1))d^{l}_{M0}(h)$$
$$= i^l(k/r^{n-2})^{1/2} J_{l+(n-2)/2}(kr)$$
$$\times\sqrt{\Gamma(n/2)/2\pi^{n/2}}\, Y_{lM}(\mathbf{e}). \quad (3.13)$$

In the last step, we have used Eq. (2.6). $Y_{lM}(\mathbf{e})$ are the hyperspherical harmonics in n dimensions.[1] The integers m_i of the set $M=(m_1,\dots,m_{n-2})$ are related by $l\geqslant m_1\geqslant m_2\geqslant\cdots\geqslant m_{n-3}\geqslant|m_{n-2}|\geqslant 0$. With Eq. (2.12) the normalization

$$\int d\mathbf{r}\, \Psi_{klM}(\mathbf{r})\Psi^{*}_{k'l'M'}(\mathbf{r}) = \delta(k-k')\delta_{ll'}\delta_{MM'} \quad (3.14)$$

is shown immediately.

Performing the integration in (3.12) by using formula #6.6332 of Ref. 7, we obtain

$$K(\mathbf{r}_b,\mathbf{r}_a;T)$$
$$= \frac{m}{i\hbar T}(r_a r_b)^{(2-n)/2} \exp\left\{\frac{im}{2\hbar T}(r_b^2+r_a^2)\right\}$$
$$\times \sum_{l=0}^{\infty} I_{l+(n-2)/2}\left(\frac{m r_a r_b}{i\hbar T}\right)$$
$$\times \sum_{M} \frac{\Gamma(n/2)}{2\pi^{n/2}} Y_{lM}(\mathbf{e}_b) Y^{*}_{lM}(\mathbf{e}_a). \quad (3.15)$$

For $n=3$, (3.15) reduces to the result of Peak and Inomata.[8]

Finally we would like to mention that the k integration can be directly performed in Eq. (3.9) via (3.5), leading to the original result of Feynman,[3]

$$K(\mathbf{r}_b,\mathbf{r}_a;T) = \left(\frac{m}{2\pi i\hbar T}\right)^{n/2} \exp\left\{\frac{im}{2\hbar T}|\mathbf{r}_b-\mathbf{r}_a|^2\right\}. \quad (3.16)$$

IV. DISCUSSION

In the present work we have applied the expansion in zonal spherical functions, developed in Ref. 1, to the path integration over the Euclidean group in n dimensions. The technique has been explicitly demonstrated for the n-dimensional free particle. Our result for the free particle coincides with that obtained via the Gaussian path integration, as expected. However, in the present approach the application of group theoretical methods has been extended to include the radial path integration. Until now only the angular path integration over rotation groups had been considered. Now we may conclude that the complete path integral treatment can be incorporated in the formalism of Ref. 1. Here the Euclidean group has been considered. However, the same technique may be applied, for example, to the path integration over the pseudo-Euclidean group $T^n \otimes SO(n-1,1)$ in n dimensions.

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